of the boundary) of the functional of attached mass in the x_1 direction, provided that the total area of the system of bodies and their attached mass in the x_2 direction are given.

The proof of condition (20) shows that the inequality becomes an equality on bodies of such form, that $\partial^2\chi/\partial x_i\partial x_j$ are constant in S^+, or by virtue of the continuous character of χ and its first derivatives, when the following relations hold on the boundary:

$$\varphi(x_1, x_2, x_3) = a^l x_l^2 + l_i;\ x = (x_1, x_2, x_3) \in \Gamma_i$$
$$\partial\varphi/\partial n = 2a^l x_l n^l,\ \ n = (n^1, n^2, n^3),\ \ i = 1,\ 2, \ldots m$$

which are also generated by the inverse problem of the theory of elasticity dealing with the optimization of the state of stress of the homogeneous, isotropic, linearly elastic space S^- with cavities loaded at infinity along the axes by the forces q_l $(l = 1, 2, 3)$. By the optimization we mean the control of the form of the boundary resulting in attainment of the least possible local Mises criterion, i.e. the maximum of the second invariant deviator of the stress tensor in S^-. The functions $\partial\chi/\partial x$ have the meaning of elastic displacements of the points of the medium along the axes $2a_l = (Q - 2q_l)/q_l$, $Q = q_1 + q_2 + q_3$, and the constants C_l remain undetermined. For such a boundary M and A are reduced simultaneously to diagonal form.

Unlike the plane case /5/, the actual determination of the boundary at $m > 1$ is very complicated. In the axisymmetric variant $(q_1 = q_2, \mu_{11} = \mu_{22})$ we propose in /7/ a non-linear integral equation in coordinates of the points lying on the meridian section of the boundary, as functions of the arc lengths, and gives the results of its numerical solution.

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ON CERTAIN FEATURES OF THE FLOWS OF VISCOUS COMPRESSIBLE FLUIDS IN CYLINDRICAL PIPES*

V.N. BELONENKO and O.YU. DINARIEV

The flow of a viscous compressible fluid in cylindrical pipes when there is volume viscosity /1/ is studied. The process is assumed to be barotropic, as is the case when, for example, heat emission can be neglected or when the fluid has high thermal conductivity. The problem of the correct boundary conditions for the system of defining equations is discussed. The problem of the flow of fluid with Tate's equation of state is solved using the method of separation of variables. Proofs of the existence and uniqueness of the solutions of the ordinary differential equations obtianed are given. The asymptotic behaviour of the velocity as the volume viscosity increased is studied. The coefficients of the volume and shear viscosity are assumed to be constant everywhere.

1. We shall consider, side by side, the plane and the three-dimensional problem of a one-dimensional steady flow in a cylindrical region enclosed between fixed walls. The defining system of equations (Navier-Stokes, continuity and state) is reduced to

$$\rho u u_{,x} = [-p + \zeta u_{,x}]_{,x} + \eta_s(\partial_x^2 + \Delta_j)u \qquad (1.1)$$
$$0 = \mathrm{grad}_j[-p + \zeta u_{,x}]$$
$$(\rho u)_{,x} = 0,\ p = p(\rho);\ \zeta = \eta_v + {}^1/_3\eta_s$$

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Here u is the flow velocity along the Ox axis, and η_v and η_s are the volume and shear viscosity coefficients. The parameter j takes two values, 1 and 2, and these correspond to flow between planes and flow in a pipe. For $j = 1$ $\Delta_1 = \partial_y^2$, $\mathrm{grad}_1 = \partial_y$ and all functions are assumed smooth in the closed region

$$G_1 = \{(x, y) \in R^2 \mid x \in [0, L], y \in [-a, a]\}$$

up to its boundary. We impose the condition of adhesion $u|_{y=\mp a} = 0$ on the velocity.

For $j = 2$ $\Delta_2 = (\partial_y^2 + \partial_z^2)$, $\mathrm{grad}_2 = (\partial_y, \partial_z)$ and all functions in the region

$$G_2 = \{(x, y, z) \in R^3 \mid x \in [0, L], (y, z) \in D_2\}$$

are assumed smooth up to its boundary: $D_2 \subset R^2$ is an arbitrary plane region with a piecewise smooth boundary, and the boundary condition $u|_{[0,L]\times\partial D_2} = 0$ is imposed on the velocity.

Let us find the function $h = -p + \zeta u_{,x}$. We see from (1.1) that h depends only on x and is equal to the pressure with its sign changed at points lying on the fixed flow boundaries with abscissa x.

Using the function h we can rewrite (1.1) thus

$$\rho u u_{,x} = h_{,x} + \eta_s (\partial_x^2 + \Delta_j) u \qquad (1.2)$$
$$u_{,x} = \zeta^{-1} (h + p); \ (\rho u)_{,x} = 0, \ p = p(\rho)$$

Specifying the external pressure drop as natural, given e.g. by the conditions $p|_{x=0} = p_0$, $p|_{x=L} = p_1$ $(p_0 > p_1)$, leads to a contradiction as shown by the following lemma.

Lemma. In the case of a viscous compressible fluid whose flow is governed by the equations (1.2) and conditions of adhesion, the relation $p|_{x=0} = p_0 = \mathrm{const}$ leads to $u \equiv 0$.

Proof. Since u vanishes on the walls, we have

$$h(0) + p|_{x=0} = h(0) + p_0 = \zeta u_{,x}|_{x=0} = 0$$

and this yields $u_{,x}|_{x=0} \equiv 0$, $\Delta_j u_{,x}|_{x=0} \equiv 0$.

Differentiation with respect to x yields the following relations from (1.2):

$$\rho u u_{,xx} = h_{,xx} + \eta_s (\partial_x^2 + \Delta_j) u_{,x} \qquad (1.3)$$
$$u_{,xx} = \zeta^{-1} (h_{,x} + p_{,\rho}\rho_{,x})$$
$$u_{,xxx} = \zeta^{-1} (h_{,xx} - p_{,\rho\rho}\rho_{,x}^2 + p_{,\rho}\rho_{,xx})$$
$$\rho_{,x} u + \rho u_{,x} = 0, \ \rho_{,xx} u + 2\rho_{,x} u_{,x} + \rho u_{,xx} = 0$$

We shall consider all term of (1.3) at $x = 0$, denoting them by the superscript $^\circ$ Then we have from the first equation of (1.3)

$$\rho^\circ u^\circ u_{,xx}^\circ = h_{,xx}^\circ + \eta_s u_{,xxx}^\circ$$

Since $u_{,xx}$ and $u_{,xxx}$ vanish at the walls, we have $h_{,xx}^\circ = 0$ and

$$\rho^\circ u^\circ u_{,xx}^\circ = \eta_s u^\circ_{,xxx} \qquad (1.4)$$

We further have

$$u^\circ u_{,xx}^\circ = h_{,x}^\circ u^\circ \zeta^{-1}, (u^\circ)^2 u_{,xxx}^\circ = p_{,\rho}^\circ (u^\circ)^2 \rho_{,xx}^\circ \zeta^{-1} = -p_{,\rho}^\circ \rho^\circ u^\circ u_{,xx}^\circ \zeta^{-1} = -p_{,\rho}^\circ \rho^\circ u^\circ h_{,x}^\circ \zeta^{-2}$$

Then from (1.4) it follows that

$$\rho^\circ (u^\circ)^3 h_{,x}^\circ = -\eta_s p_{,\rho}^\circ \rho^\circ u^\circ h_{,x}^\circ \zeta^{-1}$$

The last equation yields the following relation for the points at which $u^\circ \neq 0$ (and hence for the whole set $\{x = 0\} \cap G_j$):

$$\rho^\circ (u^\circ)^2 h_{,x}^\circ \zeta = -p_{,\rho}^\circ \rho^\circ h_{,x}^\circ \eta_s$$

But then $h_{,x}^\circ = 0$, $u_{,xx}^\circ = 0$.

Thus when $x = 0$ obeys the j-dimensional Laplace equation in some bounded region with zero boundary conditions. This yields $u|_{x=0} \equiv 0$, since $(\rho u)_{,x} = 0$, we have $u = 0$ in the whole set G_j.

The lemma show that in the case of barotropic, one-dimensional steady motions of a viscous compressible fluid between fixed walls, neither the pressure, nor the density can be assumed constant across the flow. Therefore we define the pressure drop by the following conditions:

$$h(0) = -p_0, \ h(L) = -p_1 \qquad (1.5)$$

2. We take, as the equation of state, the relation

$$p = p_0 + A(1 - \rho_0/\rho)$$

describing the behaviour of a wide class of liquids /2/.

Then in system (1.2) we can separate the variables

$$u = e^{\nu x} U, \ \rho = e^{-\nu x} R, \ h = A e^{\nu x} - A - p_0$$

where U and R are independent of x and obet the equations

$$\nu R U^2 = A\nu + \eta_s(\nu^2 + \Delta_j) U, \quad \zeta\nu U = A(1 - \rho_0/R) \tag{2.1}$$

The parameter ν is found from condition (1.5)

$$\nu = L^{-1}\ln(1 + \Delta p/A)$$

We can eliminate R from (2.1), thus obtaining a non-linear differential equation for U. It is best to write $U = A\nu^{-1}\zeta^{-1}\varphi$ and consider the problem for the function φ

$$\begin{gathered}\eta_s(\Delta_j + \nu^2)\varphi - A\rho_0\zeta^{-1}\varphi^2(1-\varphi)^{-1} + \zeta\nu^2 = 0\\ \varphi|_{y=\pm a} = 0 \ (j = 1); \ \varphi|_{\partial D_2} = 0 \ (j = 2)\end{gathered} \tag{2.2}$$

We shall assume for simplicity that in the three-dimensional case the fluid moves in a circular pipe of radius a, and the flow is axisymmetric. This means that $\varphi = \varphi(r)$, $\Delta_2 = r^{-1}\partial_r r\partial_r$, $r = \sqrt{y^2 + z^2}$.

Let us now change to a system of units in which $a = 1$, $\eta_s = 1$, $\rho_0 = 1$. Then problem (2.2) will transform to

$$\begin{gathered}\Delta_j\varphi = F(\varphi) \equiv A\zeta^{-1}\varphi^2(1-\varphi)^{-1} - \zeta\nu^2 - \nu^2\varphi\\ \varphi(\pm 1) = 0 \ (j = 1); \ \varphi(1) = 0, \ \varphi'(0) = 0 \ (j = 2).\end{gathered} \tag{2.3}$$

Theorem. Problem (2.3) has a unique solution, φ is a convex function and its maximum value satisfies the inequality

$$1 > \varphi_{\max} > 1 - A[\zeta(\zeta\nu^2 - 2j)]^{-1} \tag{2.4}$$

provided that $\zeta\nu^2 > 2j$.

Proof. Equation $F(\varphi) = 0$ has two roots: φ_1 and φ_2, and $\varphi_2 < 0 < \varphi_1 < 1$. At large ζ

$$\varphi_1 = 1 + O(\zeta^{-2}), \quad \varphi_2 = -\zeta + O(1)$$

Fig.1 shows a graph of $F = F(\varphi)$.

Let $j = 1$, and let us inspect the plane pattern of Eq.(2.3) in the region $\varphi \in [0, 1[$ (Fig.2). We have a saddle-type singularity at the point $M = (\varphi_1, 0)$. Two separatrices exist, passing through M and intersecting the $O\varphi'$ axis. The curves, together with the $O\varphi'$ axis, bound the region of possible flows. The trajectories filling this region represent the solutions of (2.3). Every trajectory is characterized by the abscissa of its intersection with the $O\varphi$ axis where the function $\varphi = \varphi(y)$ attains its maximum value $\varphi_{\max} = \varphi_m$. Since

$$dy = \pm\left(-2\int_\varphi^{\varphi_m} F(s)\,ds\right)^{-1/2} d\varphi$$

to satisfy the boundary conditions we must require that

$$W_1(\varphi_m) \equiv \int_0^{\varphi_m}\left(-2\int_\varphi^{\varphi_m} F(s)\,ds\right)^{-1/2} d\varphi = 1 \tag{2.5}$$

Let us study the behaviour of the function $W_1 = W_1(\varphi_m)$ in the interval $[0, \varphi_1]$. To do this we introduce the notation

$$f(\varphi) = -F(\varphi), \quad H(\varphi) = \int_0^\varphi f(s)\,ds, \quad H_m = H(\varphi_m)$$

Let $g = g(H)$ be the inverse of $H = H(\varphi)$. Then

$$W_1(\varphi_m) = \int_0^{H_m}[f \circ g(H)]^{-1}(H_m - H)^{-1/2}dH = 2^{-1/2}\int_0^1[f \circ g(H_m\xi)]^{-1}(1-\xi)^{-1/2}H_m^{1/2}d\xi$$

It is clear that $W_1(0) = 0$ and $W_1(\varphi_m) \to +\infty$, if $\varphi_m \to \varphi_1$.

We shall show that $dW_1(\varphi_m)/d\varphi_m > 0$. To do this, it is sufficient to note that

$$d[(f \circ g(H_m\xi))^{-1}H_m^{1/2}]/dH_m = 2^{-1}H_m^{-1/2}f^{-3}(f^2 - 2Hf') > 0$$

Indeed, the function $Q(H) \equiv [f \circ g(H)]^2 - 2Hf' \circ g(H)$ has the obvious properties $Q(0) > 0$ and $Q'(H) = -2f^{-1}Hf'' \circ g(H) > 0$, and this implies that $Q(H) > 0$ in the interval in question.

Thus we have shown that $W_1(\varphi_m)$ increases monotonically in the interval $[0, \varphi_1[$ from 0 to $+\infty$. Therefore Eq.(2.4) has a unique solution and this implies the existence and uniqueness of the solution of problem (2.3) for $j = 1$.

Let us consider the case $j = 2$. Introducing a new function $q = r^2$ and a new independent parameter $\tau = \ln r$, we can rewrite the equation (2.3) in the form of an autonomous system (a dot denotes differentiation with respect to τ)

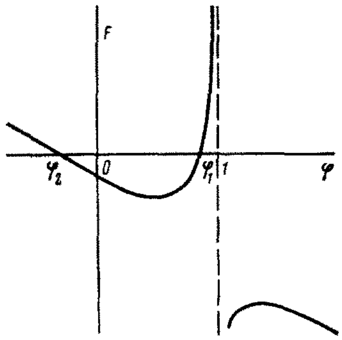

Fig.1

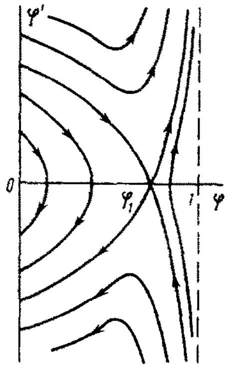

Fig.2

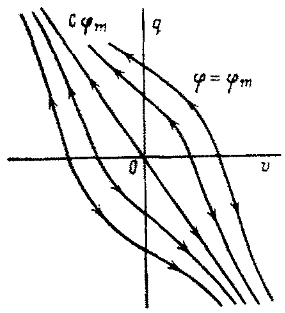

Fig.3

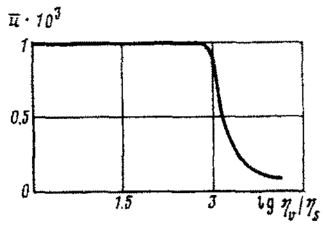

Fig.4

$$q^{\cdot} = 2q, \ \varphi^{\cdot} = v, \ v^{\cdot} = qF(\varphi) \tag{2.6}$$

The solution of problem (2.3) is represented by a trajectory of the system (2.6) lying in the set $\{(q, \varphi, v) \mid \varphi \in [0, \varphi_1[\}$ and passing through the points $A_0 = (0, \varphi_{m0}, 0)$ and $B_0 = (1, 0, v_0)$. System (2.6) has a singular curve, namely the $O\varphi$ axis. The behaviour of the trajectories of the system near this curve at $\varphi_m \in [0, \varphi_1[$ is shown qualitatively in the form of a projection onto the Ovq plane in Fig.3. We see that a unique trajectory C_{φ_m}, exists emerging from the point $A = (0, \varphi_m, 0)$ into the physical region $\{q > 0, v < 0\}$.

Let $q = q(\varphi_m, \tau)$, $\varphi = \varphi(\varphi_m, \tau)$, $v = v(\varphi_m, \tau)$ be the equation of this trajectory. Eliminating τ, we can obtain the functions $q = q(\varphi_m, \varphi)$ and $v = v(\varphi_m, \varphi)$. We have the relation

$$q(\varphi_m, \varphi) = 2^{1/2} \int\limits_{\varphi}^{\varphi_m} \left(- \int\limits_{l}^{\varphi_m} q(\varphi_m, s) F(s)\, ds \right)^{-1/2} q(\varphi_m, l)\, dl \tag{2.7}$$

Let us introduce the function $W_2(\varphi_m) = q(\varphi_m, 0)$. The equation

$$W_2(\varphi_m) = 1 \tag{2.8}$$

determines φ_{m0} and hence the solution of problem (2.3). Clearly, $W_2(0) = 0$.

Further, using relation (2.7) we can show by means of lengthy although simple manipulations, that $\partial q(\varphi_m, \varphi)/\partial\varphi_m > 0 \ (\varphi \in [0, \varphi_m])$ and hence $dW_2(\varphi_m)/d\varphi_m > 0$. The proof of this fact is basically analogous to the proof of the inequality $dW_1(\varphi_m)/d\varphi_m > 0$ and is therefore not given here. It is also clear that $W_2(\varphi_m) \to +\infty$ as $\varphi_m \to \varphi_1$ (otherwise making φ_m in (2.7) tend to φ_1, would produce a contradiction). Therefore, Eq.(2.7) has a unique solution and this implies the existence and uniqueness of the solution of problem (2.3) for $j = 2$.

We note that

$$\varphi(y) = -(1-y)\int\limits_0^y d\xi F(\varphi(\xi)) - \int\limits_y^1 (1-\xi) F(\varphi(\xi))\, d\xi \quad (j = 1)$$

$$\varphi(r) = -\ln\frac{1}{r}\int\limits_0^r \xi\, d\xi F(\varphi(\xi)) - \int\limits_r^1 \xi\, d\xi F(\varphi(\xi)) \ln\frac{1}{\xi} \quad (j = 2)$$

Estimating the terms of the identities with the help of the inequalities $0 \leqslant \varphi \leqslant \varphi_{m0} < 1$, $\varphi^2 (1-\varphi)^{-1} < (1-\varphi_{m0})^{-1}$, we obtain the inequality (2.4).

Thus we have shown that in case of the motion of a viscous compressible fluid with the Tate equation of state, a convex velocity profile forms at high densities between the plates or in a pipe of circular cross-section, whose amplitude increases exponentially in the downstream direction. The velocity has the form $u = \bar{f}\varphi$, where $\bar{f} = AL(1 + \Delta p/A)^{x/L} [(\eta_v + {}^1/_3\eta_s) \ln (1 + \Delta p/A)]^{-1}$, and

$$\varphi = \varphi(y/a)\ (j = 1),\ \varphi = \varphi(r/a)\ (j = 2)$$

is a dimensionless function taking values in the interval $[0, 1[$.

A quantitative estimate of the influence of the volume viscosity on the fluid flow was obtained by solving (2.3) numerically for the case when $j = 1$, $A\rho_0 a^2 \eta_s^{-2} = 1, aL^{-1} \ln(1 + \Delta p/A) = 0.04$, and for various values of η_v/η_s. The results obtained were used to draw a graph of $\bar{u} = (\eta_v/\eta_s + {}^1/_3)^{-1}\varphi_{max}$ versus $\lg \eta_v/\eta_s$ Fig.4. The function differs from the maximum velocity by a dimensional multiplier independent of η_v. In addition, graphs were drawn of the function $\varphi(y/a)$, characterizing the flow velocity profile for various values of η_v/η_s (Fig.5) where the curves *1, 2, 3* correspond to the values $\eta_v/\eta_s = 100, 400, 1000$. Figs.4 and 5 illustrate the assertion proved in Sect.2 that then the volume viscosity increases and other parameters are kept constant, φ_{max} tends to unity and the relation $u_{max} \simeq \bar{f}$ is satisfied asymptotically for the maximum velocity of flow.

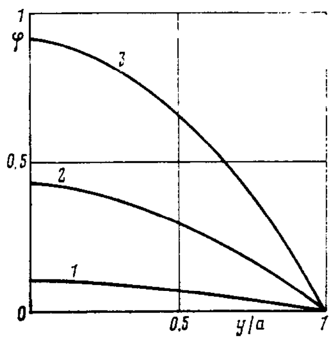

Fig.5

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SHOCK WAVES IN AN ISOTHERMAL GAS IN THE PRESENCE OF REACTION FORCES*

YU.N. GORDEYEV, N.A. KUDRYASHOV and V.V. MURZENKO

One-dimensional isothermal gas flow taking into account reaction forces which depend linearly on the velocity is considered. Problems of gas flow with and without convective terms are formulated. Their analytic and numerical solutions are obtained, and the possibility of obtaining shock waves reflected within the medium is indicated.

The flows in question arise when a gas is filtered through porous media, during its passage along pipes and major cracks, when porous bodies move in gaseous media, and in a number of technological processes /1, 2/. A system of equations describing the motion of a gas taking frictional forces into account if given in /3, 4/. The general types of systems of quasilinear equations were studied in /5/.

1. Formulation of the problem. A system of equations describing a one-dimensional isothermal gas flow with resistance forces linear with respect to the velocities, has the form

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) = 0 \tag{1.1}$$

$$\rho\left(\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u\right) = -\frac{\partial}{\partial x}P - au, \quad P = c^2\rho$$

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